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NUMERICAL STUDY OF ZAKHAROV EQUATION AS A MODEL FOR NONLINEAR WAVE-WAVE INTERACTION

S K Dadhich[1], Arun Kumar[2], Ramdayal Pankaj[3]
Department of Mathematics, Maharishi Arvind International Institute of Technology Ranpur Kota (Raj.), India[1]
Department of Mathematics, Government College, Kota (Raj.), India[2]
Department of Mathematics, J.N.V. University, Jodhpur (Raj.), India[3]

ABSTRACT:
In the paper, the coupled 1D Zakharov equation is considered as the model equation for wave-wave interaction in ionic media. A finite difference scheme is derived for the model equations. A new six point scheme, which is equivalent to the multi-symplectic integrator, is derived. The numerical simulation is also presented for the model equations.

KEYWORDS: Zakharov equation, Multi-symplectic scheme; Energy conservation; Six-point scheme

1. INTRODUCTION:

Wave-wave interaction is an important problem for both physical and mathematical reasons. Physically, the wave-wave interaction or the wave collisions are common phenomena in science and engineering for both solitary and non-solitary waves. Mathematically solitary wave collision is a major branch of nonlinear wave interaction in ionic media. Its application can be found in many areas of mathematics and physics, including nonlinear optics and plasma physics [1,11,12]. Much work has been done on interactions in large array of physical systems. Various interaction scenarios such as transmission, reflection, annihilation, trapping, creation of solitary waves and even mutual spiralling have been reported. However in their numerical simulations, in order to keep the accuracy, there are many constraints. Moreover they neglect many properties of the system, such as energy conservation, momentum conservation, etc. Several attempts were done to solve the above mentioned coupled 1D nonlinear Schrödinger system and it is solved both analytically and numerically.

Recently, specification has been paid to multi-symplectic geometry [2–4,8]. Bridge and Reich introduced the concept of multi-symplectic integrator in the form of finite difference scheme for some conservative PDEs [5,9,10]. The theoretical results indicated that it is a strictly local concept and can be formulated in the form of finite difference scheme. Thus the multi-symplectic integrator has excellent local invariant conserving properties [13,15,16]. The CNLS system has multi-symplectic structure; therefore we can apply this approach to obtain multi-symplectic integrator in difference equations form. In the paper, we discretize the system with finite difference schemes to show the multi-symplectic structure of CNLS system. We prove the advantage of the multi-symplectic structure of CNLS system by numerical simulations. In Section 2, We derived a six point difference scheme which is equivalent to multi-symplectic integrator for coupled nonlinear Schrödinger system. In Section 3, we investigate the conservation property of coupled nonlinear Schrödinger system. In Section 4, numerical simulations are reported to couple nonlinear Schrödinger system.

The Coupled Partial Differential Equations of Zakharov equation type is given by

$$iE_t + E_{xx} - nE = 0,$$

$$n_{tt} - n_{xx} - |E|_{xx}^{2} = 0$$

Here, $E(x,t)$ is the slowly varying envelope of the high frequency field, and n is the density of the media or ions in media. These Zakharov equations can be approximated by Nonlinear Schrödinger Equation.

Where $E = p(x,t) + iq(x,y), \eta = \mu(x,t) + i\xi(x,y)$

We have

$$q_t - p_{xx} = (q\xi - p\mu)$$
$$p_t + q_{xx} = (p\xi + q\mu)$$
$$\mu_{tt} - \mu_{xx} - \left((p)_{xx}^2 + (q)_{xx}^2\right) = 0$$
$$\xi_{tt} - \xi_{xx} = 0$$

Introducing the canonical momenta

$$p_x = b, q_x = a, \mu_x = d, \zeta_x = c$$
$$\mu_t = e, \xi_t = f, g = \left(p^2\right)_x, h = \left(q^2\right)_x$$

The above system can be written in the following form

$$Kz_t + Lz_x = \nabla_z S(z)$$

is multi-symplectic in nature with the state variables

$$z = \left(p, q, b, a, \mu, \xi, d, c, e, f, g, h, p^2, q^2\right)^T \in R^{14}$$

The system is multi-symplectic in the sense that K is a skew-symmetric matrix representative of the t direction and L is a skew-symmetric matrix representative of the x direction. S represents a Hamiltonian function

$$q_t - p_{xx} = (\xi q - p\mu)\ ;\ p_t + q_{xx} = (p\xi + q\mu)\ ;\ \ p_x = b\ ;\ q_x = a$$

$$\mu_{tt} - \mu_{xx} - \left((p)_{xx}^2 + (q)_{xx}^2\right) = 0$$

$$\xi_{tt} - \xi_{xx} = 0\ ;\ \mu_x = d\ ;\ \xi_x = c;\ \mu_t = e\ ;\ \xi_t = f$$

$$\left(p^2\right)_x = g;\ \left(q^2\right)_x = h$$

and we get

$$q_t - b_x = (\xi q - p\mu)\ ;\ p_t + a_x = (p\xi + q\mu)$$

$$p_x = b;\ q_x = a\ ;\ e_t - d_x - (g_x + h_x) = 0\ ;\ f_t - c_x = 0$$

$$\mu_x = d\ ;\ \xi_x = c\ ;\ \mu_t = e\ ;\ \xi_t = f\ ;\ \left(p^2\right)_x = g;\ \left(q^2\right)_x = h$$

So that

$$\nabla_z S(z) = \left(kp, kq, b, a, sp, sq, d, c, e, f, g, h, p^2, q^2\right)^T$$

$$kp = (\xi q - p\mu)\ ;\ kq = (p\xi + q\mu); p_x = b\ ;\ q_x = a\ ;$$

$$sp = 0\ ;\ sq = 0;\ \mu_x = d\ ;\ \xi_x = c; \mu_t = e;$$

$$\xi_t = f\ ; \left(p^2\right)_x = g ; \left(q^2\right)_x = h$$

$$K=\begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\quad
L=\begin{bmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&-1&-1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

Using midpoint difference scheme to discretize multi-symplectic CNLS system, we can get

$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}}{\Delta t}-\frac{b_{l+1}^{n+1/2}-b_{l}^{n+1/2}}{\Delta x}=\left(\hat{\xi}\hat{q}-\hat{p}\hat{\mu}\right) \tag{1}$$

$$\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}}{\Delta t}+\frac{a_{l+1}^{n+1/2}-a_{l}^{n+1/2}}{\Delta x}=\left(\hat{p}\hat{\xi}+\hat{q}\hat{\mu}\right) \tag{2}$$

$$\frac{p_{l+1}^{n+1/2}-p_{l}^{n+1/2}}{\Delta x}=b_{l+1/2}^{n+1/2} \tag{3}$$

$$\frac{q_{l+1}^{n+1/2}-q_{l}^{n+1/2}}{\Delta x}=a_{l+1/2}^{n+1/2} \tag{4}$$

$$\frac{e_{l+1/2}^{n+1}-e_{l+1/2}^{n}}{\Delta t}-\frac{d_{l+1}^{n+1/2}-d_{l}^{n+1/2}}{\Delta x}-\left(\frac{g_{l+1}^{n+1/2}-g_{l}^{n+1/2}}{\Delta x}+\frac{h_{l+1}^{n+1/2}-h_{l}^{n+1/2}}{\Delta x}\right)=0 \tag{5}$$

$$\frac{f_{l+1/2}^{n+1}-f_{l+1/2}^{n}}{\Delta t}=\frac{c_{l+1}^{n+1/2}-c_{l}^{n+1/2}}{\Delta x} \tag{6}$$

$$\frac{\mu_{l+1}^{n+1/2}-\mu_{l}^{n+1/2}}{\Delta x}=d_{l+1/2}^{n+1/2} \tag{7}$$

$$\frac{\xi_{l+1}^{n+1/2}-\xi_{l}^{n+1/2}}{\Delta x}=c_{l+1/2}^{n+1/2} \tag{8}$$

$$\frac{\mu_{l+1/2}^{n+1}-\mu_{l+1/2}^{n}}{\Delta t}=e_{l+1/2}^{n+1/2} \tag{9}$$

$$\frac{\xi_{l+1/2}^{n+1}-\xi_{l+1/2}^{n}}{\Delta t}=f_{l+1/2}^{n+1/2} \tag{10}$$

$$\frac{\left(p^{2}\right)_{l+1/2}^{n+1}-\left(p^{2}\right)_{l+1/2}^{n}}{\Delta x}=g_{l+1/2}^{n+1/2}$$

$$\frac{\left(q^{2}\right)_{l+1/2}^{n+1}-\left(q^{2}\right)_{l+1/2}^{n}}{\Delta x}=h_{l+1/2}^{n+1/2}$$

$$\hat{\mu}=\mu_{l+1/2}^{n+1/2},\hat{q}=q_{l+1/2}^{n+1/2},\hat{p}=p_{l+1/2}^{n+1/2},\hat{\xi}=\xi_{l+1/2}^{n+1/2}$$

From (1), (3), we eliminate b. So we can get

$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}+q_{l+3/2}^{n+1}-q_{l+3/2}^{n}}{\Delta t}-\frac{2\left(p_{l+2}^{n+1/2}-2p_{l+1}^{n+1/2}+p_{l}^{n+1/2}\right)}{\left(\Delta x\right)^{2}}$$
$$=\left(\xi_{l+1/2}^{n+1/2}q_{l+1/2}^{n+1/2}-\mu_{l+1/2}^{n+1/2}p_{l+1/2}^{n+1/2}\right)+\left(\xi_{l+3/2}^{n+1/2}q_{l+3/2}^{n+1/2}-\eta_{l+3/2}^{n+1/2}p_{l+3/2}^{n+1/2}\right) \tag{11}$$

From (2), (4), we eliminate a. So we can get

$$\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}+p_{l+3/2}^{n+1}-p_{l+3/2}^{n}}{\Delta t}+\frac{2\left(q_{l+2}^{n+1/2}-2q_{l+1}^{n+1/2}+q_{l}^{n+1/2}\right)}{\left(\Delta x\right)^{2}}$$

$$=\left(\xi_{l+1/2}^{n+1/2}p_{l+1/2}^{n+1/2}+\mu_{l+1/2}^{n+1/2}q_{l+1/2}^{n+1/2}\right)+\left(\xi_{l+3/2}^{n+1/2}p_{l+3/2}^{n+1/2}+\mu_{l+3/2}^{n+1/2}q_{l+3/2}^{n+1/2}\right) \tag{12}$$

we eliminate e and d. Also g&h.So we can get

$$\left(\frac{\mu_{l+1}^{n+2}-2\mu_{l+1}^{n+1}+\mu_{l+1}^{n}}{(\Delta t)^2}\right)-\left(\frac{\mu_{l+2}^{n+1}-2\mu_{l+1}^{n+1}+\mu_{l}^{n+1}}{(\Delta x)^2}\right)-\left(\frac{(p^2)_{l+2}^{n+1}-2(p^2)_{l+1}^{n+1}+(p^2)_{l}^{n+1}}{(\Delta x)^2}+\frac{(q^2)_{l+2}^{n+1}-(q^2)_{l+1}^{n+1}+(q^2)_{l}^{n+1}}{(\Delta x)^2}\right)=0 \quad (13)$$

we eliminate c and f so we can get

$$\left(\frac{\xi_{l+1}^{n+2}-2\xi_{l+1}^{n+1}+\xi_{l+1}^{n}}{(\Delta t)^2}\right)-\left(\frac{\xi_{l+2}^{n+1}-2\xi_{l+1}^{n+1}+\xi_{l}^{n+1}}{(\Delta x)^2}\right)=0 \quad (14)$$

Multiply (12) with *i*. And adding Eq. (11) then we can get

$$i\frac{\left(E_{l+2}^{*\,n+1}+2E_{l+1}^{*\,n+1}+E_{l}^{*n+1}\right)-\left(E_{l+2}^{*\,n}+2E_{l+1}^{*\,n}+E_{l}^{*n}\right)}{2\Delta t}$$
$$-\frac{\left(E^{*n}_{l+2}+E^{*n+1}_{l+2}\right)-2\left(E^{*n}_{l+1}+E^{*n+1}_{l+1}\right)+\left(E_{l}^{*n}+E_{l}^{*n+1}\right)}{(\Delta x)^2}$$
$$=-\frac{1}{4}\left(\left(E_{l+1/2}^{*\,n}+E_{l+1/2}^{*\,n+1}\right)\left(\eta_{l+1/2}^{*\,n}+\eta_{l+1/2}^{*\,n+1}\right)+\left(E_{l+3/2}^{*\,n}+E_{l+3/2}^{*\,n+1}\right)\left(\eta_{l+3/2}^{*\,n}+\eta_{l+3/2}^{*\,n+1}\right)\right) \quad (15)$$

Conjugating Eq. (15)

$$i\frac{\left(E_{l+2}^{n+1}+2E_{l+1}^{n+1}+E_{l}^{n+1}\right)-\left(E_{l+2}^{n}+2E_{l+1}^{n}+E_{l}^{n}\right)}{2\Delta t}$$
$$+\frac{\left(E_{l+2}^{n}+E_{l+2}^{n+1}\right)-2\left(E_{l+1}^{n}+E_{l+1}^{n+1}\right)+\left(E_{l}^{n}+E_{l}^{n+1}\right)}{(\Delta x)^2}$$
$$=\frac{1}{4}\left(\left(E_{l+1/2}^{n}+E_{l+1/2}^{n+1}\right)\left(\eta_{l+1/2}^{n}+\eta_{l+1/2}^{n+1}\right)+\left(E_{l+3/2}^{n}+E_{l+3/2}^{n+1}\right)\left(\eta_{l+3/2}^{n}+\eta_{l+31/2}^{n+1}\right)\right) \quad (16)$$

Multiply (14) with *i*. And adding Eq. (13) then we can get

$$\frac{\left(\eta_{l+1}^{n+2}-2\eta_{l+1}^{n+1}+\eta_{l+1}^{n}\right)}{(\Delta t)^2}-\frac{\left(\eta_{l+2}^{n+1}-2\eta_{l+1}^{n+1}+\eta_{l}^{n+1}\right)}{(\Delta x)^2}-\left(\frac{\left|E_{l+2}^{n+1}\right|^2-2\left|E_{l+1}^{n+1}\right|^2+\left|E_{l}^{n+1}\right|^2}{(\Delta x)^2}\right)=0 \quad (17)$$

4. NUMERICAL SIMULATION

In this section, we present the numerical result of the CNLS system using the multi-symplectic integrator. As for conserving quantities, we focus on monitoring the energy conserving properties of the multi-symplectic integrator.

Now we consider the CNLS system

$$iu_t+u_{xx}+\left(|u|^2+\beta|v|^2\right)u=0$$
$$iv_t+v_{xx}+\left(|v|^2+\beta|u|^2\right)v=0,$$

with the initial value

$$u(x,0)=\sqrt{2}r_1\operatorname{sech}\left(r_1x+\tfrac{1}{2}D_0\right)e^{tV_0x/4},$$
$$v(x,0)=\sqrt{2}r_2\operatorname{sech}\left(r_2x-\tfrac{1}{2}D_0\right)e^{-tV_0x/4}$$

From [11,12], we know, when $\beta=1$ and $\beta=0$, the CNLS system is the integrable system. Here we consider the interaction of two waves with the initial condition (25). We take the time step $\Delta t=0.02$ and a space step $\Delta x=0.2$, $-30\le x\le 30$, $D_0=25$, $r_1=r_2=1$ and $V_0=1$. In Fig. 1, the computation is done for $0\le t\le 48$. We can see after the colliding of the two soliton waves, they move forward in the same direction and with the same velocity as before

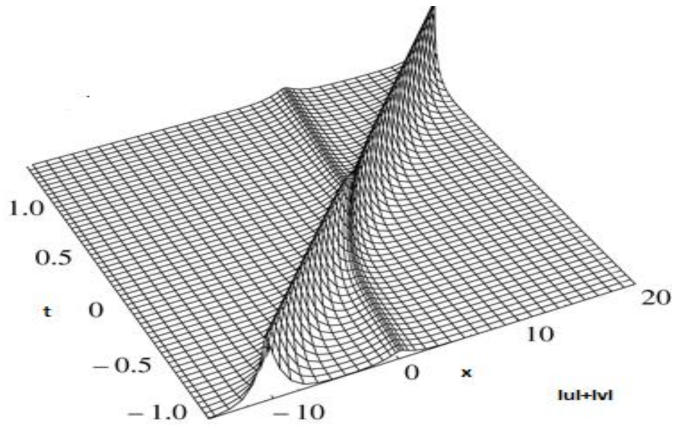

Fig. 2. Simulation results of the interaction of the two waves with β = 1

CONCLUSIONS

In this paper, the multi-symplectic formulation for the coupled 1D nonlinear Schrödinger system is presented. Numerical experiments are also reported. We observe that the multi-symplectic scheme well simulates the evolution of the solitons and preserves energy conservation well. It has advantage for the long time computing accuracy and preserving the energy conservation property.

REFERENCES

1. A. Hasegawa, Optical Solitons in Fibers, Springer-Verlag, Berlin, 1989.
2. T.J. Bridge, Multi-symplectic structures and wave propagation, Math. Proc. Cambridge Philos. Soc. 121 (1997) 147–190.
3. J.E.Marsden, S. Shkoller, Multisymplectic geometry, covariant Hamiltonians and water waves,Math. Proc. Cambridge Philos. Soc. (1999) 125.
4. B.F. Feng, T. Mitsui, A finite difference method for the Kortewegde Vries and the Kadomtsev–Petviashvili equations, J. Comput. Appl. Math. 90 (1998) 95–116.
5. T.J. Bridges, S. Reich, Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplectic, Phys. Lett. A 284 (2001) 184–193.
6. T.T. Liu,M.Z. Qin,Multisymplectic geometry and multi-symplectic Preissman scheme for the KP equation, JMP 43 (8) (2002) 4060–4077.
7. K. Feng, M.Z. Qin, The symplectic methods for computation of Hamiltonian equations, in: Y.L. Zhu, B.Y. Guo (Eds.), Proc. Conf. on Numerical Methods for PDE's, in: Lecture Notes in Math., Springer, Berlin, 1987, pp. 1–37.
8. M.Z. Qin,W.J. Zhu, Construction of symplectic schemes for wave equation via hyperbolic function sinh*(x)*, cosh*(x)* and tanh*(x)*, Comput. Math. Appl. 26 (8) (1993) 1–11.
9. T.J. Bridges, Multisymplectic structures and wave propagation, Math. Proc. Cambridge Philos. Soc. 121 (1997) 147–190.
10. Bansal K.K., Sharma M.K.[2015] A Comparative Study of Reliability Analysis of a Non-Series Parallel Network. *International Journal of Education and Science Research Review Vol.2(6)*
11. Sharma M.K. Bansal K.K. [2015] Fuzzy Analysis Of Shortest Job First. *International Journal of Engineering Research & Management Technology Vol.2(3)*
12. Sharma M.K. Bansal K.K. [2015] A Comparative Study of Reliability Analysis of a Non-Series Parallel Network. *International Journal of Education and Science Research Review Vol.2(6)*
13. S. Reich, Multi-symplectic Runge–Kutta methods for Hamiltonian wave equations, JCP 157 (2000) 473–499.
14. J. Yang, Multi Solitons perturbation theory for the Manakov equations and its applications to nonlinear optics, Phys. Rev. E 59 (2) (1999) 2393.
15. M.J. Ablowitz, H. Segur, Solitions and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
16. P.F. Zhao, M.Z. Qin, Multisymplectic geometry and multi-symplectic Preissman scheme for the KdV equation, J. Phys. A Math. Geom. 33 (2000) 3613–3626.
17. Y.S. Wang, M.Z. Qin, Multisymplectic geometry and multisymplectic schemes for the nonlinear Klein–Gordon equation, J. Phys. Soc.Japan 70 (3) (2001) 653–661.
18. Y.J. Sun, M.Z. Qin, Construction of multi-symplectic scheme of any finite order for modified wave equation, JMP 41 (11) (2000) 7854–7868.
19. J.B. Chen, M.Z. Qin, Y.F. Tang, Symplectic and multi-symplectic method for the nonlinear Schrödinger equation, Comput. Math. Appl. 43 (2002) 1095–1106.